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IS THE ECONOMY LIKE THE WEATHER? INFLATION, NONLINEARITY AND THE MATHEMATICS OF CHAOS

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How should the recent instability of oil prices be understood? By analogy to fluid dynamics, erratic prices could be seen as "economic turbulence." The mathematics currently used to model turbulence, chaos theory, is shown to be relevant to price theory. Even the simplest supply-demand interactions may become chaotic, in the presence of nonlinear price adjustment processes. In more complete economic models, speculative or other destabilizing behavior may lead to complex chaotic patterns. For econometric forecasting, chaos would imply that failures of prediction are inevitable. For policymakers, chaotic price movements would require new regulatory approaches.

1. INTRODUCTION

Do market prices tend toward a stable equilibrium? Conventional economic theory, backed by impressive mathematical models, assures us that the answer is normally "yes", with few qualifications or hesitations.

Yet some important prices, such as the price of petroleum, have appeared quite volatile over the last decade. The instability of oil prices has had disastrous effects on oil-using sectors of the economy. In many cases the problem is not that oil prices are too high or too low, but rather that they are never stable.

In the auto industry, for example, the 1973-74 oil crisis caused a consumer stampede toward smaller cars. But just as the manufacturers finished the expensive process of retooling to make small cars, the real price of oil dropped. So during 1975-78, big cars and vans were back in demand; result, another expensive retooling. Then the second oil crisis of 1979-80 caused a panicked switch back to small cars - followed again by oil glut and drift back toward bigger cars over the last few years. Literally billions of dollars have been wasted, and hundreds of thousands of labor-years idled, as the auto industry has tried to respond to the shifting signals of the market.

How should such instability be understood in theory? Oil price jumps are often viewed as exogenous shocks, caused by the events of Middle Eastern politics. But this is too simple. The Middle East is unfortunately the scene of a continuous flow of political crises. Why didn't the Israeli invasion of Lebanon in 1982, and all the other crises, cause oil shocks? The answer is the OPEC oil ministers know that they can only get away with big price hikes at times of high and rising demand. Such times came in 1973 and again in 1979, but clearly not in 1982.

But this is to say that even OPEC oil price determination is in large part rational profit-maximizing behavior - which should be endogenous in a comprehensive theory of the inflationary process.

This paper is an exploration into the theory of unstable prices. Part 2 compares price theory and fluid dynamics and suggests the possibility of "economic turbulence." The mathematics currently used in studies of turbulence, chaos theory, is introduced in Part 3. An example of chaos in elementary price theory is presented in Part 4. Part 5 surveys the more complex patterns which appear in higher-dimensional chaotic models, and Part 6 discusses the dilemmas which chaos poses for econometric forecasting and for regulatory policy.

2. MECHANICS, HISTORY OR METEOROLOGY?

Where should we look for the mathematics of unstable prices? Three analogies may be contrasted, comparing price movements to mechanics, history, or fluid dynamics.

Starting with the earliest mathematical models, a century ago, it has often been assumed that price movements are analogous to a mechanical process: like billiard balls on a flat surface or planets orbiting the sun, prices are thought to move in precise, predictable paths. A study by Mirowski details the importance of a very explicit, mechanical analogy in the writings of Walras, Jevons and Fisher [9]. Mathematically, mechanical models involve linear differential equations, which have satisfyingly simple, stable phase portraits under a wide variety of assumptions.

But in physics, the existence of mechanical laws of motion seems to be confirmed by the predictive power of such laws. In economics, despite the mathematical sophistication of current theories, the ability to predict price movements in advance remains rather minimal.

In response, institutionalists, and likely most of the general public, view economic events in historical, rather than mechanical, terms. While important generalizations can be made about prices, these generalizations are so embedded in political, social and institutional forces that they do not allow useful mathematical formulation. This may be thought of as a "null hypothesis" for the entire project of mathematical modelling.

A third approach, neither historical nor mechanical, can also be suggested. It is implicit in the very language of economists, in the common statement that resources flow to new uses in response to price changes. If the economy is usually in equilibrium, as many models assume, then the characteristics of the resource flows are unimportant. But if, for whatever reason, the world is out of equilibrium for long periods of time, then analysis of the market cannot be based on the nature of a never-reached equilibrium. What must be examined, instead, is the nature of the ongoing flow, the constant movement of resources in response to changing price signals.

Thus the third analogy: is economics analogous to fluid dynamics? The study of aerodynamics and hydrodynamics (which are very similar) has produced extensive analysis of flow problems; can some of it be used in economics? One crucial question, in particular, is the distinction between smooth and turbulent flow. Turbulence means eddies, swirls, or rapids in a stream. It happens in air as well as water; a flag on a flagpole flaps in the wind because of turbulence in the air swirling around the flagpole. Is the auto industry flapping in the wind of turbulent oil prices?

More broadly, the most important turbulent real-world system is the weather. The turbulence analogy, then, suggests that the economy should be thought of as resembling the weather. Neither history nor mechanics, but rather meteorology, may be the discipline to which economics should be compared.

3. CHAOS THEORY

Understanding turbulence is one of the great unsolved problems in mathematical physics today. The basic equations of motion for a fluid, the Navier-Stokes equations, have been known for over a century. When the Reynolds number of a fluid (roughly the ratio of velocity to viscosity) is low enough, there are unique, smooth solutions to these equations (for instance, see [5], pp. 30-31).

As the Reynolds number is increased, however, even under laboratory conditions, the transition from smooth to turbulent flow does not always come at the same point. Moreover, the patterns of swirls and disturbances created, after the very early stages of turbulence at least, do not remain the same from one trial to the next. Gross, visible features of the flow seem to depend on uncontrollably slight variations in initial conditions.

Over the past ten years it has become common to accept the hypothesis that the unruly behavior of turbulence reflects the existence of "chaotic" solutions to the equations of motion. Chaotic solutions are ones which wander around erratically; they do not converge toward stationary or periodic time paths, nor do they diverge toward infinity. Instead they bounce around in a bounded but utterly aperiodic manner which cannot be described by any familiar mathematical functions.

The mathematics of chaos crops up elsewhere in the natural sciences as well. The other most important area is in biological population studies, where difference equations describing the year-to-year changes in an animal population sometimes have chaotic solutions [8]. Chaos has also been detected in chemical reactions, lasers, electrical circuits, and perhaps even in the erratic reversals, every few million years or so, of the earth's magnetic field [3]. These are not stochastic processes; they are not displaying intrinsic indeterminacy in the manner of quantum mechanics. Each step of a chaotic process is governed by a deterministic equation of motion, yet the result is a time path that "looks" stochastic.

Chaotic solutions do not appear in <u>linear</u> systems of differential or difference equations; the solutions to linear systems are well-known and well-behaved. Either they converge to a stable equilibrium, display periodic oscillation, or diverge. But some of the simplest <u>nonlinear</u> systems can lead to chaos. The most extensively studied case is the difference equation

$$(1) \qquad x_{t+1} = ax_t(1-x_t)$$

(in which it is assumed that $1 \leq a \leq 4$ and $0 < x_0 < 1$, since outside those ranges x_t heads rapidly for either 0 or $-\infty$.)

Certainly (1) is among the simplest possible nonlinear systems. And for low enough values of the parameter a, it has correspondingly simple dynamics. As long as a is between 1 and 3, x converges toward a unique stationary solution, regardless of the initial value. For any a between 3 and about 3.57, x converges toward a periodic solution; however, as a rises the period of the stable solution increases, approaching infinity as a approaches 3.57.

As a passes beyond 3.57, equation (1) enters the realm of chaos. Slight variations in a or x_0 often produce large qualitative changes in the time path of x. For any value of a greater than about 3.83, there are periodic points of period k (that is, values of x_t which first repeat themselves k time periods later) for every positive integer k; and there are an uncountable number of initial values which are not periodic points, and which lead to time paths that do not converge toward any periodic solution [7, 8].

This result is not specific to equation (1); many similar nonlinear difference equations, including some that occur in biological population models, have similar dynamics. In general, any difference equation $x_{t+1} = f(x_t)$, with f continuous, which ever produces four consecutive values satisfying

(2) $$x_{t+2} > x_{t+1} > x_t \geq x_{t+3}$$

(or the opposite condition, with all inequality signs reversed) will exhibit the kind of chaos found in (1) when $a > 3.83$: points of period k for every integer k, and an uncountable number of paths which are not even asymptotically periodic. The article which proved this result [7] is titled "Period Three Implies Chaos", since any periodic solution to a single difference equation with period 3 must satisfy either condition (2) or its opposite.

In one of the first uses of chaos theory in economics, Day has shown the Solow's neoclassical growth equation can be doctored a bit to produce the symptoms of condition (2), and hence to exhibit chaotic growth paths. He also demonstrates graphically how a tiny (0.1%) variation in initial conditions can cause gross difference in qualitative behavior within 20 to 30 time periods [2].

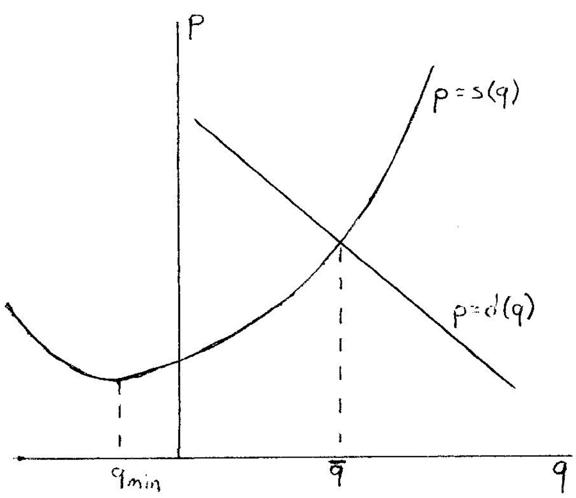

Figure 1. Supply and demand curves. (No economic meaning is attached to the extension of the supply curve to negative quantities; however, the value of q_{min} is used in the text below.)

Yet Day's model seems difficult to apply to the real world. He suggests that the time period of his model should be thought of as a generation ([2], p. 409). Thus the mathematically interesting behavior shown in his graphs after 20 to 30 time periods will presumably occur after 20 to 30 generations, provided that all else remains equal.

4. FROM COBWEBS TO CHAOS

Day's model, and the few other published examples of chaos in economics, seem somewhat esoteric. Yet a simple example will show that chaos can occur in the core of economic theory, in the supply and demand for a single product. Consider the time-honored supply and demand curves (Figure 1). What dynamic behavior can they lead to?

If both supply and demand respond immediately to disequilibrium, then the dynamics are trivial. Any normal-looking supply and demand curves produce immediate convergence to equilibrium, and the market can change only when the curves shift.

But suppose that demands reacts immediately, while supply adjustments happen with a one-period lag. Think of the product as petroleum, and assume that oil producers will sell any quantity desired, but will adjust prices in response to last period's sales. Letting p_t and q_t represent price and quantity at time t, and s(.) and d(.) the supply-price and demand-price adjustment functions, the market relations can be expressed as

(3a) $$p_t = s(q_{t-1})$$

(3b) $$p_t = d(q_t)$$

or, since d is invertible,

(3c) $$q_t = d^{-1}(s(q_{t-1}))$$

Economic theory has very little to say about the exact functional forms of s and d. The traditional approach takes the Taylor series expansion of s and d at $\bar{q}$, the equilibrium - and assumes that all terms above the first order can be ignored [10]. Then equations (3) become

(4a) $$p_t = s_0 + s_1 q_{t-1}$$

(4b) $$p_t = d_0 - d_1 q_t$$

or, eliminating p_t as in equations (3),

(4c) $$q_t = \frac{d_0 - s_0}{d_1} - \frac{s_1}{d_1} q_{t-1}$$

(where s_0, s_1, d_0, d_1 are positive constants from the Taylor series expansions of s and d).

Equation (4c) generates the standard cobweb cycle. Its behavior is determined by the value of $c = s_1/d_1$, the ratio of the slopes of s and d at $\bar{q}$. There are three possibilities:

i) if $c < 1$, q_t converges to $\bar{q}$;
ii) if $c = 1$, there are periodic solutions of period 2; and
iii) if $c > 1$, q_t (and p_t) diverge farther and farther from equilibrium over time.

Of these possibilities, iii) seems obviously unrealistic, and ii) occurs only for one highly improbable exact parameter value. So it is easy to infer that i) is the normal state of affairs. Mathematical theory alone appears to lead to an empirically significant conclusion - the only plausible alternative is that the market is stable and converges to equilibrium.

The same three alternatives, and the same plausibility argument, apply to much more complex linear systems as well. As long as all the equations are linear, a system of differential or difference equations leads either to divergence, which is unrealistic as an economic model; to periodic oscillation, which occurs only for improbable parameter values; or to convergence. A sleight-of-hand trick has made the possibility of instability vanish before our eyes - the card we were watching is suddenly no longer on the table.

It is not hard to explain this particular magic trick. The card was palmed and whisked away when the higher-order terms of the Taylor series expansions were discarded, and the general equations (3) became the linear equations (4). This linear approximation can be justified only if the analysis is confined to very small fluctuations around equilibrium, or if the true functional forms of s and d in (3) are very close to linearity. Neither condition applies to the market for petroleum: supply and demand clearly fluctuate far from equilibrium, and significant nonlinearities are visible in OPEC's supply behavior. The same is true, if less dramatically, in other real-world markets.

Suppose, then, that the supply curve is nonlinear, and that just one more term of the Taylor series must be included to obtain a useful approximation. That is, the supply curve is closer to being a parabola, as drawn in Figure 1, than it is to a straight line. In this case equations (3) become

(5a) $$p_t = s_0 + s_1 q_{t-1} + s_2 (q_{t-1})^2$$

(5b) $$p_t = d_0 - d_1 q_t$$

or, as before,

(5c) $$q_t = \frac{d_o - s_o}{d_1} - \frac{s_1}{d_1} q_{t-1} - \frac{s_2}{d_1}(q_{t-1})^2$$

(where the constants are defined as before, and s_1 will be positive whenever q_{min} in Figure 1 is negative; note that s_1 is no longer the slope of the supply curve).

Equation (5c) can be converted into the form of equation (i) by a linear transformation, $q_t = ux_t + v$, for suitably chosen constants u and v. Such a linear transformation does not alter the qualitative phase portrait of the system: convergence to a stationary or periodic solution, chaos, or divergence are all preserved by linear changes of variables. Thus (5c), the nonlinear cobweb equation, can display the full range of bizarre behavior discovered in (1). When (5c) is transformed into (1), the critical parameter a of (1) becomes

(6a) $$a = 1 + \frac{\sqrt{(s_1+d_1)^2 + 4s_2(d_o - s_o)}}{d_1}$$

Let c again mean the ratio of the slopes of s and d at $\bar{q}$ (which is no longer equal to s_1/d_1). Then both of the following expressions are equivalent to (6a):

(6b) $$a = 2 + c$$

(6c) $$a = 2s_2(\bar{q} - q_{min})/d_1$$

Using (6b), and the description of the solutions to (1) presented above, the behavior of (5c) may be summarized as follows:

i) if $c < 1$ (that is, if $a < 3$), q_t converges to $\bar{q}$;
ii) if $1 \leq c < 1.57$, q_t converges to a stable periodic solution;
iii) if $1.57 \leq c \leq 2$, there are bounded, chaotic solutions;
iv) if $c > 2 (a > 4)$, q_t diverges to $\pm \infty$.

This is more complex than the dynamics of (4c), the linear cobweb equation; the results are the same only when $c < 1$ or $c > 2$. For c between 1 and 2, the linear approximation predicts divergence, while the inclusion of one nonlinear term reveals the possibility of either stable cycles or chaotic fluctuations.

While (6b) facilitates comparison with the linear approximation, (6c) may help to visualize the economic conditions which lead to instability. The greater the curvature of the supply curve (bigger s_2), and the greater the price-elasticity of demand (smaller d_1), the more unstable the market will be. (The other factor, $\bar{q} - q_{min}$, is harder to interpret.) And in fact, the market for petroleum has turned out to have more nonlinearity of supply and more price-elasticity of demand than anyone would have guessed in pre-energy-crisis days.

5. "DIMENSION THREE PERMITS CHAOS"

Most interesting economic behavior requires more complex models, which cannot be reduced to a single equation like (3c). And the analysis of nonlinear multiple-equation models is very difficult. The appealing "period three implies chaos" theorem is quite specific to the single-equation case; counter-examples to it can be created in models with two variables and two difference equations. Yet although the simplicity of condition (2) is lost, chaotic solutions are actually more abundant in multiple-equation models.

In one of the few multiple-equation chaotic models in economics, Benhabib and Day have shown that chaos can ensue when consumers exhibit "tricyclic" preferences: the activities preferred in the first and second time periods do not overlap, but the activities preferred in the third period include all those chosen in the preceding two periods [1]. A person on vacation might play tennis on the first day, read on the couch and recover from tennis on the second day, but do either on the third. In general, tricyclic preferences do not seem realistic; they appear to have been tailored to fit a particular mathematical theorem, a too-narrow generalization of "period three implies chaos" to multiple-equation systems.

A better generalization of "period three implies chaos" is available; it suggests somewhat broader economic applications. Let X_t be an n-dimensional vector, and consider the equation

$$(7) \qquad X_{t+1} = F(X_t)$$

If F is differentiable and has an equilibrium vector Z = F(Z), then (7) has chaotic solutions near Z whenever both of the following hold:

i) The equilibrium at Z is locally unstable; that is, any slight disturbance in Z leads to a time path which initially moves further away from Z. Mathematically, all eigenvalues of the Jacobian of F have absolute value greater than 1 in a neighborhood of Z; and

ii) A time path from some nearby point eventually returns to Z. Mathematically, there is an X_0 near Z and a positive integer m such that $F^m(X_0) = Z$, and the Jacobian of F^m has nonzero determinant at X_0.

In summary, any system of difference equations is chaotic in the neighborhood of a reachable but locally unstable equilibrium. For reasons which are not at all immediately clear, these conditions are said to reduce to "period three implies chaos" in the one-dimensional case [6].

Are these conditions economically realistic? Notice that speculative behavior creates local instability of the type called for in the theorem. If individuals and firms expect present rates of change, rather than present levels, of prices to persist, they will behave in a manner which will amplify any price fluctuations. (This phenomenon is discussed, in a different mathematical framework, in the recent literature on "price bubbles.")

Destabilizing speculation is involved in the auto/oil problem, as outlined above. The panic-stricken demand for small cars at times of oil crisis is not rational if the current price of gasoline is expected to continue. It may, however, be rational if current rates of change in prices are extrapolated into the future. And the more that people act as if price increases will continue, the more it will come true.

Thus far the discussion of chaos has dealt with discrete-time, difference equation models. In continuous-time models, chaotic solutions cannot occur with only one or two differential equations. However, in a system of three or more nonlinear differential equations chaos becomes possible. This does not reduce the importance of chaos: physical turbulence, for example, is a three-dimensional phenomenon, requiring at least three equations; and interesting economic problems generally involve at least three interdependent variables.

The fact that, in continuous-time models, "dimension three permits chaos" may have far-reaching implications for current economic theories. Many economists formulate models involving an indeterminate large number, n, of commodities or actors; then, for the sake of simplicity, the implications of the model are worked out for the special case of n = 2. The appeal of two-dimensional models is obvious: on a plane, graphs can be drawn, tangency conditions can be examined, phase diagrams can be created, etc. But this simplification assumes away one of the most interesting possibilities of instability. A mathematics text often cited by economic theorists emphasizes that there are powerful two-dimensional stability theorems with <u>no</u> higher-dimensional analogues ([5] , pp. 239-240, 314.)

6. BUTTERFLIES AND ECONOMETRICS

Chaotic time paths are sensitively dependent on initial conditions, so that the inevitable tiny errors in empirical data may cause major errors in any estimates based on the data. This has become known as the "butterfly problem" in theories of meteorology: even if all the initial conditions for weather prediction were known with absolute certainty, a butterfly fluttering its wings could create new air currents and thus upset all long-range forecasts. In other words, long-range prediction is impossible in the realm of chaos. So, too, is estimation of parameters from chaotic time series data.

Suppose that the underlying dynamics of the economy are chaotic. What would that imply for econometric forecasting? In the short run, the estimates would not look too bad; almost anything, even chaos, can be approximated with a linear model in the short run. Despite the butterfly problem, forecasts of tomorrow's weather are quite often accurate. In the longer run, linear approximations to chaotic patterns will always diverge, requiring ad hoc readjustments which then again allow reasonable predictions for a while. Since chaos is often more volatile than solutions to linear equations, the linear approximations will miss particularly often in predicting turning points.

In short. linear approximations to chaotic processes would produce roughly the pattern of problems which econometric forecasting models of the U.S. economy have today. The hypothesis that chaos exists in the world would imply that the forecasting failures are fundamental and incurable. No matter how often a model is corrected to take account of last year's errors, it is bound to fail again.

Returning to the choice of analogies presented to Part 1 - mechanics, meteorology or history - the search for mathematical laws of motion in a volatile market economy may lead to one of two problematical alternatives. "Mechanical," usually linear, models offer definite, describable, nonchaotic solutions - at the cost of having to explain many, perhaps increasingly many, fluctuations as external, "historical" shocks. On the other hand, "meteorological" models offer simple, endogenous, deterministic explanations of erratic volatility - at the cost of being unable to estimate the parameters of the models from empirical data.

In the natural sciences, the discovery of chaos in a model is often taken as a sign that detailed analysis is impossible at the level of abstraction. A frequent response is to "jump up" to a higher level and study the long-run average behavior of the chaotic process.

For some purposes, such as understanding fluids moving through a pipeline, or the ecology of animal populations, long-run averages may be quite useful. But in understanding both the economy and the weather, we are stuck with the intrinsic importance of events on a human time scale ("real time," as they say in the computer business). We are forced to remain interested in next month's thunderstorms and next year's oil prices, and gain little information from twenty-year averages of rainfall or (as Keynes observed) from "long run" economic outcomes.

In conclusion, some speculation about the policy implications of chaotic prices may be in order. As explained in the introduction, the volatility of oil prices is costly to the auto industry and other oil-dependent sectors of the economy. This free-market volatility cannot be described as "efficient" in any meaningful sense of the term. So it appears at first glance that controlling prices at almost any level would be preferable to uncontrolled fluctuation. The auto industry could adjust to high oil prices and small cars, or to low oil prices and big cars, so long as it is known which it will be for several years in a row.

But aside from the political obstacles to controlling oil prices, it would be difficult to guess the correct price in advance. A wrong guess would lead to a mounting shortage or surplus, and to the collapse of controls.

A more modest approach might concentrate on simply smoothing the erratic time path of prices. Limits could be placed on the allowable rate of change of oil prices in either direction - similar to the "dirty float" in the European currency system, for example. Whether this or another approach is adopted, chaotic price fluctuation suggests that something is wrong with the market mechanism, and calls for creative rethinking of regulatory strategy.

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